The Biduals of Dual Banach Bimodules

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Abstract

We give two necessary and sufficient conditions in order that a bidual of a dual Banach bimodule is a dual Banach bimodule and the left and right actions are weakly* continuous. We see that a C^*-algebra satisfies the sufficient condition, so that the bidual of a C^*-algebra is a Banach *-algebra and the multiplication is separately weakly* continuous.

Let E, A and B be three Banach spaces and, A and B act on E as linear spaces, that is, for each $\xi \in E, x \in A$ and $y \in B$, $x\xi \in E$ and $\xi y \in E$ are defined and the mappings $A \times E \ni (x, \xi) \mapsto x\xi \in E$ and $B \times E \ni (y, \xi) \mapsto \xi y \in E$ are bilinear. E is called a Banach A-B bimodule if these bilinear mappings are continuous and $x(\xi y) = (x\xi)y$ for every $\xi \in E, x \in A$ and $y \in B$. When A and B are Banach algebras, usually the following conditions are assumed: $x(y\xi) = (xy)\xi$ for every $\xi \in E$ and $x, y \in A$ and $(\xi x)y = \xi(xy)$ for every $\xi \in E$, and $x, y \in B$. A and B are called operator domains. We call a Banach A-B bimodule E a dual Banach A-B bimodule if E is the dual of a Banach space F and the mappings $E \ni \xi \mapsto x\xi \in E$ and $E \ni \xi \mapsto \xi y \in E$ are $\sigma(E, F)$-continuous.

In Theorem 3, we give two necessary and sufficient conditions in order that the bidual E^{**} of a dual Banach A-B bimodule E is a dual Banach A^{**}-B^{**} bimodule and the left and right actions are weakly* continuous.

In Lemma 4, we see that a C^*-algebra satisfies the sufficient condition. Therefore the bidual of a C^*-algebra is a Banach *-algebra and the multiplication is separately weakly* continuous.

Proposition 1. *Let A and B be two Banach spaces and a Banach space E a Banach A-B bimodule. Then it follows that*

(i) *E^{**} is a dual Banach A^{**}-B bimodule;*

(ii) *in particular, if A is a Banach algebra and $E = A = B$, then A^{**} is a Banach algebra and a dual Banach A^{**}-A bimodule.*

Proof. For each $\xi \in E$, the linear mapping $\pi_r(\xi)\colon x \in A \mapsto x\xi \in E$ is bounded. Take the bitranspose ${}^{tt}\pi_r(\xi)\colon A^{**} \to E^{**}$ of $\pi_r(\xi)$; then we have $\|{}^{tt}\pi_r(\xi)\| = \|\pi_r(\xi)\|$. For each

$x \in A^{**}$, the linear mapping $L_x \colon E \ni \xi \mapsto {}^{tt}\pi_r(\xi)(x) \in E^{**}$ is bounded. Take the transpose ${}^t({}^tL_x|E^*)$ of the restriction ${}^tL_x|E^*$; then ${}^t({}^tL_x|E^*)$ is a bounded linear mapping of E^{**} into E^{**} and $\|{}^t({}^tL_x|E^*)\| \leq \|L_x\|$. Putting $x\xi = {}^t({}^tL_x|E^*)(\xi)$ for each $x \in A^{**}$ and $\xi \in E^{**}$, this action of A^{**} on E^{**} extends the action of A on E and the linear mapping $E^{**} \ni \xi \mapsto x\xi \in E^{**}$ is $\sigma(E^{**}, E^*)$-continuous for every $x \in A^{**}$. For each $y \in B$, the linear mapping $R_y \colon \xi \in E \mapsto \xi y \in E$ is bounded. The bitranspose ${}^{tt}R_y \colon E^{**} \to E^{**}$ is a bounded linear mapping and $\|{}^{tt}R_y\| = \|R_y\|$. Putting $\xi y = {}^{tt}R_y(\xi)$ for each $y \in B$ and $\xi \in E^{**}$, this action of B^{**} on E^{**} extends the action of B on E and the linear mapping $E^{**} \ni \xi \mapsto \xi y \in E^{**}$ is $\sigma(E^{**}, E^*)$-continuous.

For any $\xi \in E$ and $y \in B$, the mappings $A^{**} \ni x \mapsto x(\xi y) \in E^{**}$ and $A^{**} \ni x \mapsto (x\xi)y \in E^{**}$ are continuous and coincide on A. Hence these mappings coincide on A^{**}, i.e., $x(\xi y) = (x\xi)y$ for every $x \in A^{**}$. For any $x \in A^{**}$ and $y \in B$, the mappings $E^{**} \ni \xi \mapsto x(\xi y) \in E^{**}$ and $E^{**} \ni \xi \mapsto (x\xi)y \in E^{**}$ are continuous and coincide on E. Hence these mappings coincide on E^{**}, i.e., $x(\xi y) = (x\xi)y$ for every $x \in A^{**}, y \in B$ and $\xi \in E^{**}$. Therefore E^{**} is a dual Banach A^{**}-B bimodule.

(ii) Let A be a Banach algebra and put $B = E = A$. Define a multiplication in A^{**} by the left action of A^{**} on A^{**}. We shall see the associative law and then A^{**} is a Banach algebra. Notice that, for any $y \in A$, the linear mapping $A^{**} \ni x \mapsto xy \in A^{**}$ is $\sigma(A^{**}, A^*)$-continuous and, for any $x \in A^{**}$, the linear mapping $A^{**} \ni y \mapsto xy \in A^{**}$ is $\sigma(A^{**}, A^*)$-continuous. For each $y, z \in A$, the mappings $A^{**} \ni x \mapsto x(yz) \in A^{**}$ and $A^{**} \ni x \mapsto (xy)z \in A^{**}$ are continuous and coincide on A. Hence these mappings coincide on A^{**}, i.e., $x(yz) = (xy)z$ for every $x \in A^{**}$. Similarly, we have $x(yz) = (xy)z$ for all $x, y, z \in A^{**}$. □

Corollary 2. *Let E be a Banach A-B bimodule with Banach algebras A and B as operator domains. Consider the bidual A^{**} as a Banach algebra with the algebraic structure as in Proposition 1. Then E^{**} is a dual Banach A^{**}-B bimodule with Banach algebras as operator domains.*

Proof. It suffices to see the associative law: $x(y\xi) = (xy)\xi$ for every $\xi \in E$ and $x, y \in A^{**}$. For each $y \in A$ and $\xi \in E$, the mappings $A^{**} \ni x \mapsto x(y\xi) \in E^{**}$ and $A^{**} \ni x \mapsto (xy)\xi \in E^{**}$ are continuous and coincide on A. Hence these coincide on A^{**}, i.e., $x(y\xi) = (xy)\xi$ for every $x \in A^{**}$. For each $x \in A^{**}$ and $\xi \in E$, the mappings $A^{**} \ni y \mapsto x(y\xi) \in E^{**}$ and $A^{**} \ni y \mapsto (xy)\xi \in E^{**}$ are continuous and coincide on A. Hence these coincide on A^{**}, i.e., $x(y\xi) = (xy)\xi$ for every $y \in A^{**}$. Finally, for each $x, y \in A^{**}$, the mappings $E^{**} \ni \xi \mapsto x(y\xi) \in E^{**}$ and $E^{**} \ni \xi \mapsto (xy)\xi \in E^{**}$ are continuous and coincide on E. Therefore these coincide on E^{**} and so the associative law is valid. □

Let A and B be two Banach spaces and E a dual Banach A-B bimodule with a Banach space F as a predual. For each $x \in A$ and $\varphi \in F$, since the linear mapping $E \ni \xi \mapsto x\xi \in E$ is $\sigma(E, F)$-continuous, the linear form $E \ni \xi \mapsto \langle x\xi, \varphi \rangle$ is $\sigma(E, F)$-continuous and so belongs to F. We denote this by φx. Similarly, for each $y \in B$ and $\varphi \in F$, we denote the linear form $E \ni \xi \mapsto \langle \xi y, \varphi \rangle$ in F by $y\varphi$. Then F is a Banach B-A bimodule. For each $\xi \in E$ and $\varphi \in F$, we define the bounded linear forms $L_\xi \varphi$ and $R_\xi \varphi$ as follows:

$$L_\xi \varphi(y) = \langle \xi y, \varphi \rangle \quad \text{for } y \in B \quad \text{and} \quad R_\xi \varphi(x) = \langle x\xi, \varphi \rangle \quad \text{for } x \in A.$$

Theorem 3. *Let A and B be two Banach spaces and E a dual Banach A-B bimodule with a Banach space F as a predual. Then it follows that*

(i) *the following three statements are equivalent:*

(a) *E is a dual Banach A^{**}-B^{**} bimodule and, the linear mappings $A^{**} \ni x \mapsto x\xi \in E$ (resp., $B^{**} \ni y \mapsto \xi y \in E$) are continuous with respect to the $\sigma(A^{**}, A^*)$-topology (resp., the $\sigma(B^{**}, B^*)$-topology) and $\sigma(E, F)$-topology;*

(b) *for any $\varphi \in F$, the sets*

$$\{\varphi x \mid x \in A, \|x\| \le 1\} \quad \textit{and} \quad \{y\varphi \mid y \in B, \|y\| \le 1\}$$

are relatively compact in F with respect to the $\sigma(F, E)$-topology;

(c) *for any $\varphi \in F$, the balanced convex sets*

$$\{L_\xi \varphi \mid \xi \in E, \|\xi\| \le 1\} \quad \textit{and} \quad \{R_\xi \varphi \mid \xi \in E, \|\xi\| \le 1\}$$

are relatively compact in A^* (resp., B^*) with respect to the $\sigma(A^*, A^{**})$-topology (resp., the $\sigma(B^*, B^{**})$-topology);

(ii) *in particular, if A is a Banach algebra, $B = A$ and $E = A^{**}$, and the balanced convex sets*

$$\{\varphi x \mid x \in A, \|x\| \le 1\} \quad \textit{and} \quad \{x\varphi \mid x \in A, \|x\| \le 1\}$$

are relatively compact in A^* with respect to the $\sigma(A^*, A^{**})$-topology, then the multiplication in A^{**} defined by the left action of A^{**} on E coincides with the multiplication in A^{**} defined by the right action of A^{**} on E and A^{**} is a Banach algebra. Furthermore the multiplication in A^{**} is separately continuous with respect to the $\sigma(A^{**}, A^*)$-topology.

Proof. (b)$\to$(a): Assume the statement (b). For each $\xi \in E$, since the linear mapping $\pi_r(\xi) \colon A \ni x \mapsto x\xi \in E$ is bounded, the linear mapping ${}^t({}^t\pi_r(\xi)|F) \colon A^{**} \to E$ is bounded and $\|{}^t({}^t\pi_r(\xi)|F)\| \le \|\pi_r(\xi)\|$. We define the left action of A^{**} on E by $x\xi = {}^t({}^t\pi_r(\xi)|F)(x)$. Similarly, since the linear mapping $\pi_l(\xi) \colon B \ni y \mapsto \xi y \in E$ is bounded, we define the right action of B^{**} on E by $\xi y = {}^t({}^t\pi_l(\xi)|F)(y)$.

For any element x of the unit ball of A^{**}, there is a filter $\mathfrak{F}$ on the unit ball of A converging to x with respect to the $\sigma(A^{**}, A^*)$-topology. For any $\xi \in E$ and $\varphi \in F$, we have

$$\langle x\xi, \varphi \rangle = {}^t({}^t\pi_r(\xi)|F)(x) = \lim_{y,\mathfrak{F}} {}^t({}^t\pi_r(\xi)|F)(y) = \lim_{y,\mathfrak{F}} \langle y\xi, \varphi \rangle = \lim_{y,\mathfrak{F}} \langle \xi, \varphi y \rangle.$$

By the assumption, the image of $\mathfrak{F}$ under the function $A \ni y \mapsto \varphi y \in F$ has a cluster point $\psi \in F$. Hence we have $\langle x\xi, \varphi\rangle = \langle \xi, \psi\rangle$ and so the linear mapping $E \ni \xi \mapsto x\xi \in E$ is $\sigma(E, F)$-continuous. Similarly, for any $y \in B^{**}$, the linear mapping $E \ni \xi \mapsto \xi y \in E$ is $\sigma(E, F)$-continuous.

For each $y \in B$ and $\xi \in E$, the mappings $A^{**} \ni x \mapsto x(\xi y) \in E$ and $A^{**} \ni x \mapsto (x\xi)y \in E$ are continuous and coincide on A. Hence these coincide on A^{**}, i.e., $x(\xi y) = (x\xi)y$ for every $x \in A^{**}$. For each $x \in A^{**}$ and $\xi \in E$, the mappings $B^{**} \ni y \mapsto x(\xi y) \in E$ and $B^{**} \ni y \mapsto (x\xi)y \in E$ are continuous and coincide on B. Hence these coincide on B^{**}, i.e., $x(\xi y) = (x\xi)y$ for every $y \in B^{**}$. Therefore E is a dual Banach A^{**}-B^{**} bimodule.

(c)$\rightarrow$(a): Extend the left action to A^{**} and the right action of B^{**} as above. Assume the statement (c). For any element x of the unit ball of A^{**}, there is a filter $\mathfrak{F}$ on the unit ball of A converging to x with respect to the $\tau(A^{**}, A^*)$-topology. By the assumption, for any $\varphi \in F$, we have $\lim_{y,\mathfrak{F}} \sup_{\|\xi\|\leq 1} |\langle y - x, R_\xi\varphi\rangle| = 0$. We have

$$\langle x, R_\xi\varphi\rangle = \lim_{y,\mathfrak{F}}\langle y, R_\xi\varphi\rangle = \lim_{y,\mathfrak{F}}\langle y\xi, \varphi\rangle = \langle x\xi, \varphi\rangle.$$

Hence we obtain $\lim_{y,\mathfrak{F}} \sup_{\|\xi\|\leq 1} |\langle \xi, \varphi y\rangle - \langle x\xi, \varphi\rangle| = 0$. Therefore the linear form $E \ni \xi \mapsto \langle x\xi, \varphi\rangle$ is a limit of the image of $\mathfrak{F}$ under the function $A \ni y \mapsto \varphi y \in F$ with respect to the uniform topology and so belongs to F. Hence the linear mapping $E \ni \xi \mapsto x\xi \in E$ is $\sigma(E, F)$-continuous. Similarly, for any $y \in B^{**}$, the linear mapping $E \ni \xi \mapsto \xi y \in E$ is $\sigma(E, F)$-continuous, so that (a) is valid.

(a)$\rightarrow$(b): Assume the statement (a). For each $x \in A^{**}$ and $\varphi \in F$, the linear form $\varphi x\colon E \ni \xi \mapsto \langle x\xi, \varphi\rangle$ is continuous and so belongs to F. Since the mapping $A^{**} \ni x \mapsto \langle x\xi, \varphi\rangle$ is continuous, the mapping $A^{**} \ni x \mapsto \varphi x \in F$ is continuous. Since the unit ball of A^{**} is $\sigma(A^{**}, A^*)$-compact, the set $\{\varphi x \mid x \in A, \|x\| \leq 1\}$ is relatively $\sigma(F, E)$-compact. Similarly the set $\{y\varphi \mid y \in B, \|y\| \leq 1\}$ is relatively $\sigma(F, E)$-compact.

(a)$\rightarrow$(c): Assume the statement (a). For each $x \in A^{**}$ and $\varphi \in F$, the linear form $\varphi x\colon E \ni \xi \mapsto \langle x\xi, \varphi\rangle$ is continuous and so belongs to F. Since the mapping $A^{**} \ni x \mapsto \langle x\xi, \varphi\rangle$ is continuous, we have $\langle x\xi, \varphi\rangle = \langle x, R_\xi\varphi\rangle$ for every $x \in A^{**}$. Hence the mapping $E \ni \xi \mapsto R_\xi\varphi \in A^*$ is continuous with respect to the topology $\sigma(E, F)$ and $\sigma(A^*, A^{**})$-topology. Since the unit ball of E is $\sigma(E, F)$-compact, the balanced convex set $\{R_\xi\varphi \mid \xi \in E, \|\xi\| \leq 1\}$ is $\sigma(A^*, A^{**})$-compact. Similarly the balanced convex set $\{L_\xi\varphi \mid \xi \in E, \|\xi\| \leq 1\}$ is $\sigma(B^*, B^{**})$-compact.

(ii) Let A be a Banach algebra and put $B = A$ and $E = A^{**}$. Assume the sets

$$\{\varphi x \mid x \in A, \|x\| \leq 1\} \quad \text{and} \quad \{x\varphi \mid x \in A, \|x\| \leq 1\}$$

are relatively compact in A^* with respect to the topology $\sigma(A^*, A^{**})$. By Proposition 1, E is a dual Banach A-A bimodule and so a dual Banach A^{**}-A^{**} bimodule, in virtue of (i). The linear mapping $A^{**} \ni x \mapsto {}^t({}^t\pi_r(y)|A^*)(x) \in A^{**}$ is the left action of A^{**} on A^{**}

and the linear mapping $A^{**} \ni y \mapsto {}^t({}^t\pi_l(x)|A^*)(y) \in A^{**}$ is the right action of A^{**} on A^{**}. The mappings $A^{**} \ni y \mapsto {}^t({}^t\pi_r(y)|A^*)(x) \in A^{**}$ and $A^{**} \ni x \mapsto {}^t({}^t\pi_l(x)|A^*)(y) \in A^{**}$ are $\sigma(A^{**}, A^*)$-continuous and, for any $x, y \in A$, we have

$${}^t({}^t\pi_r(y)|A^*)(x) = \pi_r(y)(x) = xy = \pi_l(x)(y) = {}^t({}^t\pi_l(x)|A^*)(y).$$

Therefore, for any $x, y \in A^{**}$, we have ${}^t({}^t\pi_r(y)|A^*)(x) = {}^t({}^t\pi_l(x)|A^*)(y)$. Define the multiplication in A^{**} by

$$xy = {}^t({}^t\pi_l(y)|A^*)(x) = {}^t({}^t\pi_r(x)|A^*)(y).$$

Then the multiplication $(x, y) \in A^{**} \times A^{**} \mapsto xy \in A^{**}$ is separately continuous with respect to the $\sigma(A^{**}, A^*)$-topology. Hence the multiplication in A^{**} is an extension of the multiplication in A by continuity. Since the multiplication in A^{**} is separately continuous, it is easy to see the associative law. Therefore A^{**} is a Banach algebra. □

Let A be a Banach algebra. For each $x \in A^{**}$ and $\varphi \in A^*$, let $\tilde{L}_x\varphi$ and $\tilde{R}_x\varphi$ denote the bounded linear forms $A \ni a \mapsto \langle x, a\varphi\rangle$ and $A \ni a \mapsto \langle x, \varphi a\rangle$, respectively.

Let A be a Banach *-algebra. For each $x \in A^{**}$, put $x^*(\varphi) = \overline{x(\varphi *)}$ with $\varphi \in A^*$; then we have $x^* \in A^{**}$. Obviously the mapping $A^{**} \ni x \mapsto x^* \in A^{**}$ is conjugate linear and $\sigma(A^{**}, A^*)$-continuous. Hence the conjugate linear mapping $A^* \ni \varphi \mapsto \varphi^* \in A^*$ is $\sigma(A^*, A^{**})$-continuous. Since $(x^*)^* = x$ and $\|x^*\| \leq \|x\|$, we have $\|x^*\| = \|x\|$. Moreover, for any $x \in A^{**}$ and $\varphi \in A^*$, we have $\tilde{R}_x\varphi = (\tilde{L}_{x^*}\varphi^*)^*$. In fact, we have, for any $a \in A$,

$$\begin{aligned}\langle a, \tilde{R}_x\varphi\rangle &= \langle x, \varphi a\rangle = \overline{\langle x^*, (\varphi a)^*\rangle} = \overline{\langle x^*, a^*\varphi^*\rangle} \\ &= \overline{\langle a^*, \tilde{L}_{x^*}\varphi^*\rangle} = \langle a, (\tilde{L}_{x^*}\varphi^*)^*\rangle.\end{aligned}$$

Lemma 4. *Let A be a C^*-algebra. Then, for any $\varphi \in A^*$, the balanced convex sets*

$$\{\tilde{L}_x\varphi \mid x \in A^{**}, \|x\| \leq 1\} \quad \textit{and} \quad \{\tilde{R}_x\varphi \mid x \in A^{**}, \|y\| \leq 1\}$$

are relatively compact in A^* with respect to the $\sigma(A^*, A^{**})$-topology.

Proof. Since $\tilde{R}_x\varphi = (\tilde{L}_{x^*}\varphi^*)^*$ and the mapping $A^* \ni \varphi \mapsto \varphi * \in A^*$ is $\sigma(A^*, A^{**})$-continuous, it suffices to see that $\{\tilde{L}_x\varphi \mid x \in A^{**}, \|x\| \leq 1\}$ is relatively compact. Moreover, by Jordan decomposition, it suffices to see that for every state φ of A. For a state φ of A, let η_φ be the canonical mapping of A into the Hilbert space $\mathfrak{H}_\varphi$ associated with φ. Since $\|\eta_\varphi\| \leq 1$, take the bitranspose of η_φ. The bitranspose ${}^{tt}\eta_\varphi$ is a mapping of A^{**} into $\mathfrak{H}_\varphi^{**} = \mathfrak{H}_\varphi$. Since $\langle x, {}^t\eta_\varphi(\xi)\rangle = \langle {}^{tt}\eta_\varphi(x), \xi\rangle$ for every $x \in A^{**}$ and $\xi \in \mathfrak{H}_\varphi^*$, the mapping ${}^t\eta_\varphi$ is continuous with respect to the $\sigma(\mathfrak{H}_\varphi^*, \mathfrak{H}_\varphi)$-topology and $\sigma(A^*, A^{**})$-topology.

Since the unit ball of A is $\sigma(A^{**}, A^*)$-dense in the unit ball of A^{**}, there is a filter $\mathfrak{F}$ on the unit ball of A converging to x in the unit ball of A^{**} with respect to $\sigma(A^{**}, A^*)$-topology.

Since $a\varphi \in A^*$ for every $a \in A$, we have

$$\begin{aligned}|\langle a, \tilde{L}_x\varphi\rangle| &= |\langle x, a\varphi\rangle| = \lim_{y,\mathfrak{F}} |\langle y, a\varphi\rangle| = \lim_{y,\mathfrak{F}} |\varphi(ya)| \\ &\leq \overline{\lim_{y,\mathfrak{F}}} \varphi(yy^*)^{1/2}\varphi(a^*a)^{1/2} \leq \|\eta_\varphi(a)\|.\end{aligned}$$

Hence there is an element ξ in the unit ball of $\mathfrak{H}_\varphi^*$ such that $\xi(\eta_\varphi(a)) = (\tilde{L}_x\varphi)(a)$, so that $\tilde{L}_x\varphi = {}^t\eta_\varphi(\xi)$. Therefore the balanced convex set $\{\tilde{L}_x\varphi \mid x \in A^{**}, \|x\| \leq 1\}$ is contained in the image of the unit ball of $\mathscr{H}_\varphi^*$ under ${}^t\eta_\varphi$. Since the unit ball of $\mathscr{H}_\varphi^*$ is $\sigma(\mathscr{H}_\varphi^*, \mathscr{H}_\varphi)$-compact, the balanced convex set $\{\tilde{L}_x\varphi \mid x \in A^{**}, \|x\| \leq 1\}$ is relatively compact. It thus completes the proof. □

Proposition 5. *The bidual A^{**} of a C^*-algebra A is a unital Banach *-algebra. Furthermore the multiplication in A^{**} is separately continuous and the involution $A^{**} \ni x \mapsto x^* \in A^{**}$ is continuous with respect to the $\sigma(A^{**}, A^*)$-topology.*

Proof. By Proposition 1, A^{**} is a dual Banach A-A bimodule. By Theorem 3 and Lemma 4, A^{**} is a Banach algebra and the multiplication in A^{**} is separately continuous with respect to the $\sigma(A^{**}, A^*)$-topology.

For each $y \in A$, the mapping $A^{**} \ni x \mapsto (xy)^* \in A^{**}$ and $A^{**} \ni x \mapsto y^*x^* \in A^{**}$ are continuous and coincide on A and so coincide on A^{**}. Similarly we have $(xy)^* = y^*x^*$ for every $x, y \in A^{**}$. Hence the mapping $x \mapsto x^*$ is an involution in A^{**}, so that A^{**} is a Banach *-algebra.

A cluster point in A^{**} of an approximate identity of A is an identity of A^{**}, in virtue of the separate continuity. Consequently, A^{**} is a unital Banach *-algebra. □