Relatively Compact Subsets of the Predual of a von Neumann Algebra

Akio Ikunishi

Institute of Natural Sciences, SeiShu University, 214-8580 Japan

Abstract

Let $K$ be a relatively $\sigma(\mathcal{M}, \mathcal{M})$-compact subset of the predual $\mathcal{M}^*$ of a von Neumann algebra $\mathcal{M}$. C. Akemann proved that there exists an element $\omega \in \mathcal{M}^+$ as follows: for any positive number $\varepsilon$, there exists a positive number $\delta$ such that $|\varphi(x)| < \varepsilon$ for every $\varphi \in K$ if $\omega(x^*x + xx^*) < \delta$ and $\|x\| \leq 1$. We shall give a more distinct proof than the Akemann's that.

Let $\mathcal{M}$ be a von Neumann algebra and define the sets

$$\mathcal{M}_{11} = \{x \in \mathcal{M} \mid x^* = x, \|x\| \leq 1\}, \quad \mathcal{M}_{+1} = \{x \in \mathcal{M} \mid 0 \leq x \leq 1\}.$$

An important problem on topological property of a $\sigma(\mathcal{M}, \mathcal{M})$-compact subset of the predual is whether a $\sigma(\mathcal{M}, \mathcal{M})$-compact subset of the predual is equicontinuous in the unit ball of $\mathcal{M}$. Theorem 3 shows that this is valid for the $\sigma$-strong* topology. A convergent sequence is relatively compact. Lemma 2 shows the special case that any convergent sequence in the predual is equicontinuous in the unit ball with respect to the $\sigma$-strong* topology. By the Baire theorem, a convergent sequence in the predual is equicontinuous at some point in $\mathcal{M}_{+1}$. If $x$ and $y$ are in $\mathcal{M}_{+1}$, then we have $\|x - y\| \leq 1$ and so $(x - y)^2 \in \mathcal{M}_{+1}$. This fact solves the difficulty when one implies the equicontinuity at zero from the equicontinuity at some point. It is not difficult to see that Lemma 2 implies Theorem 3.

Lemma 1. Let $\mathcal{M}$ be a von Neumann algebra and $B$ a bounded subset of the self-adjoint portion of $\mathcal{M}$. Let $f$ be a complex valued continuous function on a bounded closed subset $I$ of $\mathbb{R}$. Assume that $\text{Sp}(x) \subset I$ for every $x \in B$. Then the function $B \ni x \mapsto f(x) \in \mathcal{M}$ is $\sigma$-strongly continuous. In particular, the functions $\mathcal{M}_{+1} \ni x \mapsto x_+ \in \mathcal{M}_{+1}$, $\mathcal{M}_{s1} \ni x \mapsto x_- \in \mathcal{M}_{+1}$ and $\mathcal{M}_{+1} \ni x \mapsto x^{1/2} \in \mathcal{M}_{+1}$ are $\sigma$-strongly continuous.

Proof. By the Weierstrass theorem, there exists a sequence $(p_n)_n$ of polynomials such that $\lim_{n \to \infty} \sup_{t \in I} |p_n(t) - f(t)| = 0$. For an element $x \in B$, considering the Gelfand representation of the commutative $C^*$-subalgebra generated by $x$ and $1$, we have

$$\|p_n(x) - f(x)\| = \sup_{\chi} |p_n(\chi(x)) - f(\chi(x))| \leq \sup_{t \in I} |p_n(t) - f(t)|,$$
where \(x\)'s are characters. Hence \((p_n(x))_n\) converges to \(f(x)\) uniformly for \(x \in B\). Since the functions \(B \ni x \mapsto p_n(x) \in \mathcal{M}\) are \(\sigma\)-strongly continuous, the function \(B \ni x \mapsto f(x) \in \mathcal{M}\) is \(\sigma\)-strongly continuous. Since the functions \([-1,1] \ni t \mapsto t \lor 0, [-1,1] \ni t \mapsto -t \land 0\) and \([0,1] \ni t \mapsto t^{1/2}\) are continuous, the functions \(\mathcal{M}_{+1} \ni x \mapsto x_+ \in \mathcal{M}_{+1}, \mathcal{M}_{+1} \ni x \mapsto x_- \in \mathcal{M}_{+1}\) and \(\mathcal{M}_{+1} \ni x \mapsto x^{1/2} \in \mathcal{M}_{+1}\) are \(\sigma\)-strongly continuous.

For an element \(\varphi \in \mathcal{M}^*\), define \([\varphi]\) by \(\frac{1}{2}(|\varphi + \varphi^*| + |i(\varphi - \varphi^*)|)\). Notice that \(\varphi(s([\varphi])s([\varphi])) = \varphi(x)\) for every \(x \in \mathcal{M}\).

**Lemma 2.** Let \((\varphi_n)_n\) be a \(\sigma(\mathcal{M},\mathcal{M})\)-convergent sequence in the predual \(\mathcal{M}\) of a von Neumann algebra \(\mathcal{M}\) with a faithful normal state. Then \((\varphi_n)_n\) is \(\sigma\)-strongly* equicontinuous at zero in the unit ball of \(\mathcal{M}\), that is, \(\lim_{x \in \mathfrak{B}} \sup_k |\varphi_k(x)| = 0\) if \(\mathfrak{B}\) is the neighbourhood filter of zero on the unit ball of \(\mathcal{M}\) with respect to the \(\sigma\)-strong* topology.

**Proof.** Replacing \(\varphi_n\) by \(\varphi_n - \lim_{n \to \infty} \varphi_n\), we may assume \(\lim_{n \to \infty} \varphi_n = 0\). Since \(\lim_{x \in \mathfrak{B}} x^* = 0\) in the \(\sigma\)-strong* topology, we may assume, without loss of generality, that \(\mathfrak{B}\) is a neighbourhood filter of zero on \(\mathcal{M}_{+1}\) with respect to the \(\sigma\)-strong topology, in virtue of Lemma 1. \(\mathcal{M}_{+1}\) equipped with the \(\sigma\)-strong topology is metrizable and complete, and hence is a Baire space. For an arbitrary fixed positive number \(\varepsilon\), define the sets
\[
S_n = \{ x \in \mathcal{M}_{+1} \mid \forall k \geq n, |\varphi_k(x)| \leq \varepsilon \}.
\]
Since \(S_n\) is \(\sigma\)-strongly closed and \(\mathcal{M}_{+1} = \bigcup_n S_n\), some \(S_n\) has an interior point \(a\). By Lemma 1, we have \(\lim_{x \in \mathfrak{B}} x^{1/2} = 0\) in the \(\sigma\)-strong topology. Since \((a^{1/2} - x^{1/2})^2 \in \mathcal{M}_{+1}\) for every \(x \in \mathcal{M}_{+1}\) and \(a = \lim_{x \in \mathfrak{B}} (a^{1/2} - x^{1/2})^2\), there is some balanced neighbourhood \(V \in \mathfrak{B}\) such that \((a^{1/2} - x^{1/2})^2 \in S_n\) for all \(x \in V\), so that \(|\varphi_k((a^{1/2} - x^{1/2})^2)| \leq \varepsilon\) for all \(k \geq n\) and \(x \in V\). Hence we have \(|\varphi_k(x) - \varphi_k(a^{1/2}x^{1/2} + x^{1/2}a^{1/2})| \leq 2\varepsilon\) for all \(k \geq n\) and \(x \in V\). For any \(k \geq n\) and \(x \in V\), we have \(|4^{-1}\varphi_k(x) - 2^{-1}\varphi_k(a^{1/2}x^{1/2} + x^{1/2}a^{1/2})| \leq 2\varepsilon\), because that \(4^{-1}x \in V\). Therefore we have \(|\varphi_k(x)| \leq 12\varepsilon\) for all \(k \geq n\) and \(x \in V\). For some \(W \in \mathfrak{B}\) such that \(|\varphi_k(x)| \leq \varepsilon\) for all \(k < n\) and \(x \in W\). Consequently, we obtain \(\lim_{x \in \mathfrak{B}} \sup_{k \geq 1} |\varphi_k(x)| = 0\) for all \(x \in V \cap W\) and so \(\lim_{x \in \mathfrak{B}} \sup_{k \geq 1} |\varphi_k(x)| = 0\).

**Theorem 3.** Let \(\mathcal{M}\) be a von Neumann algebra and \(K\) a subset of the predual \(\mathcal{M}\). In order that \(K\) is relatively compact with respect to the \(\sigma(\mathcal{M},\mathcal{M})\)-topology, it is necessary and sufficient that there exists an element \(\omega \in \mathcal{M}^*_+\) as follows: for any positive number \(\varepsilon\), there exists a positive number \(\delta\) such that \(|\varphi(x)| < \varepsilon\) for every \(\varphi \in K\) if \(\omega(x^*x + xx^*) < \delta\) and \(|x| \leq 1\), or equivalently, \(K\) is \(\sigma\)-strongly* equicontinuous on the unit ball of \(\mathcal{M}\). Hence, in any bounded set of \(\mathcal{M}\), the \(\sigma\)-strong* topology is homeomorphic to the \(\tau(\mathcal{M},\mathcal{M}_*)\)-topology.

**Proof.** Let \(\omega\) be an element of \(\mathcal{M}^*_+\) satisfying the condition in the statement; then, for any \(\varphi \in K\), we have \(|\varphi(x)| \leq \varepsilon \) if \(\omega(x^*x + xx^*) < \delta\) and \(|x| \leq 1\). If \(\varphi\) is a cluster point in \(\mathcal{M}^*_+\).
of a filter on $\overline{K}$, then we have $|\varphi(x)| \leq \varepsilon$ if $\omega(x^*x + xx^*) < \delta$ and $\|x\| \leq 1$. Hence $\varphi$ is $\sigma$-strongly* continuous on the unit ball and so $\varphi \in \mathcal{M}_*$. $\overline{K}$ is therefore $\sigma(\mathcal{M}_*, \mathcal{M})$-compact.

Suppose that $K$ is relatively compact and is included in the unit ball of $\mathcal{M}_*$. Let $\mathcal{I}$ denote the unit ball of $\mathcal{M}$. Suppose that $K$ is not $\sigma$-strongly* equicontinuous at zero in the unit ball of $\mathcal{M}$, that is, there exists a positive number $\delta > 0$ such that, for any $\omega \in \mathcal{M}_*$ and $\varepsilon > 0$, there exist $x \in \mathcal{I}$ and $\varphi \in K$ such that $\omega(x^*x + xx^*) < \varepsilon$ and $|\varphi(x)| \geq \delta$. Let $\varphi_0$ be an element of $K$ and put $\omega_0 = [\varphi_0]$; then there exist $x_1 \in \mathcal{I}$ and $\varphi_1 \in K$ such that $\omega_0(x_1^*x_1 + x_1x_1^*) < 2^{-1}$ and $|\varphi_1(x_1)| \geq \delta$. By induction, there exist sequences $(x_k)_k$ in $\mathcal{I}$ and $(\varphi_k)_k$ in $K$ such that $\omega_k(x_k^*x_k + x_kx_k^*) < 2^{-k}$ and $|\varphi_k(x_k)| \geq \delta$ for every $k \geq 1$, where $\omega_k = \sum_{i=0}^{k}[\varphi_i]$. We have $\|(k+1)^{-1}\omega_k\| \leq 2$. Put $\omega = \sum_{k=0}^{\infty}2^{-k}(k+1)^{-1}\omega_k$ and $e = s(\omega)$; then $\omega$ is faithful in $c^0$. Since $\omega_1(x_k^*x_k + x_kx_k^*) \leq \omega_k(x_k^*x_k + x_kx_k^*) < 2^{-k}$ for every $i < k$, we have

$$
\omega(x_k^*x_k + x_kx_k^*) < \sum_{i=0}^{k-1} \frac{1}{2^i} \frac{1}{i+1} \frac{1}{2^k} + 4 \sum_{i=k}^{\infty} \frac{1}{2^i} \leq \frac{10}{2^k}.
$$

Hence $(ex_ke)_k$ converges $\sigma$-strongly* to zero. There exists a subsequence $(\varphi_{i_k})_k$ of $(\varphi_k)_k$ which converges to some cluster point of $(\varphi_k)_k$ with respect to the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology.

From the construction of $\omega_k$, it follows that $\varphi_{i_k}(x_n) = \varphi_{i_k}(ex_ne)$ for every $k$ and $n$. By Lemma 2, we have $\lim_{n \to \infty} \sup_k |\varphi_{i_k}(x_n)| = \lim_{n \to \infty} \sup_k |\varphi_{i_k}(ex_ne)| = 0$, which contradicts $|\varphi_{i_k}(x_{i_k})| \geq \delta$. Therefore, $K$ is $\sigma$-strongly* equicontinuous at zero in the unit ball of $\mathcal{M}$. Consequently, the condition is necessary.

If $K$ is a compact balanced convex set of the predual, then there exists a neighbourhood $V$ of zero in the unit ball of $\mathcal{M}$ with respect to the $\sigma$-strong* topology such that $|\langle x, \varphi \rangle| \leq 1$ for all $x \in V$ and $\varphi \in K$. Hence $V \subset K^c \cap \mathcal{I}$. Therefore, in the unit ball, the $\sigma$-strong* topology is homeomorphic to the $\tau(\mathcal{M}, \mathcal{M}_*)$-topology. \hfill $\square$

**References**