On an equation with respect to derivative $\frac{dE}{dm}$ in the case of the Dirac equation

H. Uematsu

Abstract

We derive an equation with respect to derivative $\frac{dE}{dm}$ which could be useful for the analysis of the relation between energy E and mass m of the Dirac equation.

1. Introduction

We derived an equation with respect to derivative $\frac{dE}{d\mu}$ in the case of the Schrödinger equation ([2], [3], [4]), which was useful for the analysis of the relation between energy E and reduced mass μ.

We think that it is also interesting to derive a similar equation in the case of the Dirac equation. Here we mean the Dirac equation by the following equation.

$$m\alpha_0\Psi - i\sum_{j=1}^{3}\alpha_j\frac{\partial\Psi}{\partial x_j} + V\Psi = E\Psi$$

where m is the mass of the particle, E is an eigenvalue and Ψ is the four-component wave function.

The 4×4 matrices α_k $(0\le k\le 3)$ are Hermitian symmetric and satisfy the commutation relations.

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I.$$

We shall derive an equation with respect to derivative $\frac{dE}{dm}$ in this paper.

2. Results

First we consider the following operator $H(m)$ in the Hilbert space $(L^2(R^3))^4$.

For $m > 0$ we put

$$H(m) = m\alpha_0 - i\sum_{j=1}^{3}\alpha_j\frac{\partial}{\partial x_j} + V \tag{1}$$

where V is the multiplication operator by $v(x)I$.

Let us state a few assumptions.

(A1) $v(x)$ is a real locally square integrable function.

(A2) V is $H(1)$-bounded with $H(1)$-bound smaller than 1.

Noting that α_0 is a bounded operator in $(L^2(R^3))^4$, we see that $H(m)$ is a selfadjoint holomorphic family of type (A) for $m > 0$.

It follows that the eigenvalues E and the eigenfunctions of $H(m)$ are holomorphic on $m \in (0, \infty)$.

Here we have used the notions introduced by Kato([1]), so we give these definitions.

Definition 1. Let T and A be operators with the same domain space such that $\mathcal{D}(T) \subset \mathcal{D}(A)$ and

$$\|Au\| \leq a\|u\| + b\|Tu\| \qquad u \in \mathcal{D}(T)$$

where a, b are nonnegative constants.

Then we say that A is T-bounded. The greatest lower bound of all possible constants b is called the T-bound of A.

Definition 2. $T(\lambda)$ is called a selfadjoint holomorphic family of type (A) if the following conditions are satisfied.

(i) $T(\lambda)$ defined for λ in a domain D_0 is selfadjoint.

(ii) $\mathcal{D}(T(\lambda)) = \mathcal{D}$ is independent of μ.

(iii) $T(\lambda)u$ is holomorphic for $\lambda \in D_0$ for every $u \in \mathcal{D}$.

Now we are ready to state the result.

Theorem. In addition to the assumptions (A1) and (A2), suppose that $v(x) \in C^1(R^3/\{0\})$ and that there exist constants $M > 0$ and $N > 0$ such that

$$|g(x)| \leq M|v(x)| + N \tag{2}$$

where

$$g(x) \equiv \sum_{l=1}^{3} x_l \frac{\partial v}{\partial x_l}. \tag{3}$$

Then for any real number a

$$\frac{dE}{dm} = (\Psi,\ \alpha_0 \Psi) - i\frac{a}{m}\sum_{j=1}^{3}\left(\Psi(m),\ \alpha_j \frac{\partial \Psi(m)}{\partial x_j}\right) - \frac{a}{m}(\Psi,\ G\Psi) \tag{4}$$

where G is the multiplication operator by $g(x)I$.

Proof. Since Ψ is an eigenfunction of $H(m)$,

$$m\alpha_0\Psi - i\sum_{j=1}^{3}\alpha_j\frac{\partial\Psi}{\partial x_j} + V\Psi = E\Psi \tag{5}$$

where $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)$. Ψ is normalized, *i. e.*

$$(\Psi, \Psi) = \sum_{k=1}^{4}\int_{R^3}\Psi_k(x)\,\overline{\Psi_k(x)}\,dx = 1. \tag{6}$$

To manifest m dependence of Ψ and E, we denote Ψ by $\Psi(x\,;m)$ and E by $E(m)$. Then we put $\tilde{\Psi}(m)$ as follows.

$$\tilde{\Psi}(m) \equiv \Psi(m^{-a}y\,;m) \tag{7}$$

By definition $\tilde{\Psi}(m)$ satisfies the following equations.

$$\tilde{H}(m)\tilde{\Psi}(m) = E(m)\tilde{\Psi}(m) \tag{8}$$

where

$$\tilde{H}(m) \equiv m\alpha_0 - im^a\sum_{j=1}^{3}\alpha_j\frac{\partial}{\partial y_j} + \tilde{V}(m), \tag{9}$$

$$\tilde{V}(m) \equiv V(m^{-a}y).$$

Naturally we see by (6)

$$(\tilde{\Psi}, \tilde{\Psi}) = \sum_{k=1}^{4}\int_{R^3}\tilde{\Psi}_k(x)\,\overline{\tilde{\Psi}_k(x)}\,dx = m^{3a}.$$

Now we start from the following identity which holds obviously.

For any $h > 0$, we see

$$\frac{1}{h}\Big[-\Big(\tilde{H}(m+h)\tilde{\Psi}(m+h),\,\tilde{\Psi}(m)\Big) + \Big(\tilde{H}(m+h)\tilde{\Psi}(m+h),\,\tilde{\Psi}(m)\Big)$$
$$-\Big(\tilde{H}(m)\tilde{\Psi}(m+h),\,\tilde{\Psi}(m)\Big) + \Big(\tilde{H}(m)\tilde{\Psi}(m+h),\,\tilde{\Psi}(m)\Big)\Big]\,m^{-3a} = 0.$$

Moving the first term and the last term of the left side into the right, we obtain by (8)

$$\frac{E(m+h)-E(m)}{h}\Big(\tilde{\Psi}(m+h),\,\tilde{\Psi}(m)\Big)\,m^{-3a}$$
$$= \left(\tilde{\Psi}(m+h),\,\Big(\frac{\tilde{H}(m+h)-\tilde{H}(m)}{h}\Big)\tilde{\Psi}(m)\right)m^{-3a}$$
$$= \Big(\tilde{\Psi}(m+h),\,\alpha_0\tilde{\Psi}(m)\Big)\,m^{-3a} - ik(m,h)\sum_{j=1}^{3}\Big(\tilde{\Psi}(m+h),\,\alpha_j\frac{\partial\tilde{\Psi}(m)}{\partial y_j}\Big)\,m^{-3a} \tag{10}$$
$$+ \Big(\tilde{\Psi}(m+h),\,A(m,h)\tilde{\Psi}(m)\Big)\,m^{-3a}$$

where

$$k(m,h) \equiv \frac{1}{h}\Big((m+h)^a - m^a\Big),$$

$$A(m,h) \equiv \frac{1}{h}\Big(\tilde{V}(m+h) - \tilde{V}(m)\Big).$$

Since $\tilde{\Psi}(m)$ is strongly continuous, we deduce that

$$\lim_{h\to 0}\Big(\tilde{\Psi}(m+h),\, \alpha_0\tilde{\Psi}(m)\Big)\, m^{-3a} = (\Psi,\, \alpha_0\Psi), \tag{11}$$

$$\lim_{h\to 0} k(m,h)\sum_{j=1}^{3}\left(\tilde{\Psi}(m+h),\, \alpha_j\,\frac{\partial\tilde{\Psi}(\mu)}{\partial y_j}\right) m^{-3a} = \frac{a}{m}\sum_{j=1}^{3}\left(\Psi(m+h),\, \alpha_j\,\frac{\partial\Psi(m)}{\partial x_j}\right). \tag{12}$$

As for the third term in the right hand side of (10), it is essentially similar to the case of the Schrödiger equation. Therefore we see

$$\lim_{h\to 0}\Big(\tilde{\Psi}(m+h),\, A(m,h)\tilde{\Psi}(m)\Big)\, m^{-3a} = -\frac{a}{m}(\Psi, G\Psi). \tag{13}$$

In view of (11), (12) and (13) we have obtained equation (4).

Now we are going to show the following corollary concerning mass dependence of energy.

Corollary. In addition to the assumptions of the theorem, suppose that $g = -v$, then $E = cm$, where c is a constant.

Proof. It follows from (5) that

$$-i\sum_{j=1}^{3}\left(\Psi,\, \alpha_j\,\frac{\partial\Psi}{\partial x_j}\right) = E - (\Psi,\, V\Psi) - m(\Psi,\, \alpha_0\Psi). \tag{14}$$

Substituting (14) into (4), one has

$$\frac{dE}{dm} = (1-a)(\Psi,\, \alpha_0\Psi) + \frac{a}{m}E - \frac{a}{m}\Big(\Psi,\, (V+G)\Psi\Big). \tag{15}$$

Putting $a = 1$, one sees that

$$\frac{dE}{dm} = \frac{1}{m}E - \frac{1}{m}\Big(\Psi,\, (V+G)\Psi\Big). \tag{16}$$

It follows from (16) and the assumption that

$$\frac{dE}{dm} = \frac{1}{m}E. \tag{17}$$

Solving (17), we see that $E = cm$.

References

[1] **T. Kato** : Perturbation Theory For Linear Operators, Springer-Verlag Inc. (1966)

[2] **H. Uematsu** : Bulletin of the Association of Natural Science Senshu University 27 (1996) p.1–p.8

[3] **H. Uematsu** : Bulletin of the Association of Natural Science Senshu University 28 (1998) p.19–p.28

[4] **H. Uematsu** : Bulletin of the Association of Natural Science Senshu University 28 (1999) p.21–p.30