Note on maximum likelihood estimation of a linear regression model with random effects

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Abstract

In this note we consider maximum likelihood estimation of parameters in a linear regression model with random effects. The model is a generalization of ones in Reinsel (1984, 1985). The likelihood function is derived. Further we show some assumptions on covariance structure of the model in order to express maximum likelihood estimators of the parameters explicitly.

Key words: linear regression, random effect, maximum likelihood estimator, covariance structure.

1 Introduction

There are a lot of studies on growth curve models. Further random effects or repeated measurements are assumed by many authors. See references at the back of the article. Here we consider a generalization of models in Reinsel (1984, 1985).

Suppose that for $j = 1, 2, \cdots, N$ and $k = 1, 2, \cdots, n(j)$

$$\boldsymbol{Y}_{j,k} = \boldsymbol{X}_{j,k}\boldsymbol{\beta}_j + \boldsymbol{\varepsilon}_{j,k}, \tag{1}$$

where $\boldsymbol{Y}_{j,k}$'s are $p \times 1$ vectors, $\boldsymbol{X}_{j,k}$'s are $p \times q$ matrices of full rank q, $\boldsymbol{\beta}_j$'s are $q \times 1$ vectors and $\boldsymbol{\varepsilon}_{j,k}$'s are $p \times 1$ random error vectors independently and identically distributed as a normal distribution $\mathrm{N}_p\left(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon\right)$ with zero mean vector $\mathbf{0}$ and nonsingular covariance matrix $\boldsymbol{\Sigma}_\varepsilon$. Moreover, we assume that

$$\boldsymbol{\beta}_j = \boldsymbol{B}\boldsymbol{a}_j + \boldsymbol{\lambda}_j, \tag{2}$$

where $\boldsymbol{B}$ is $q \times r$ matrix, $\boldsymbol{a}_j$'s are unknown $r \times 1$ vectors and $\boldsymbol{\lambda}_j$'s are $q \times 1$ random effect vectors independently and identically distributed as a normal distribution $\mathrm{N}_q\left(\mathbf{0}, \boldsymbol{\Sigma}_\lambda\right)$ with zero mean vector $\mathbf{0}$ and nonsingular covariance matrix $\boldsymbol{\Sigma}_\lambda$.

From (1) it follows that

$$\boldsymbol{Y}_j = \begin{pmatrix} \boldsymbol{Y}_{j,1} \\ \vdots \\ \boldsymbol{Y}_{j,n(j)} \end{pmatrix}, \boldsymbol{X}_j = \begin{pmatrix} \boldsymbol{X}_{j,1} \\ \vdots \\ \boldsymbol{X}_{j,n(j)} \end{pmatrix} \text{ and } \boldsymbol{\varepsilon}_j = \begin{pmatrix} \boldsymbol{\varepsilon}_{j,1} \\ \vdots \\ \boldsymbol{\varepsilon}_{j,n(j)} \end{pmatrix},$$

where $\boldsymbol{Y}_j$'s and $\boldsymbol{\varepsilon}_j$'s are $pn(j) \times 1$ vectors and $\boldsymbol{X}_j$'s are $pn(j) \times q$ matrices. Then we have

$$\boldsymbol{Y}_j = \boldsymbol{X}_j\boldsymbol{\beta}_j + \boldsymbol{\varepsilon}_j \tag{3}$$

and

$$\boldsymbol{\varepsilon}_j \sim \mathrm{N}_{pn(j)}\left(\mathbf{0}, \boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon\right), \tag{4}$$

where $\boldsymbol{I}_{n(j)}$ is an identity matrix of order $n(j)$ and $\otimes$ denotes the Kronecker product of matrices such as

$$\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon = \begin{pmatrix} \boldsymbol{\Sigma}_\varepsilon & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_\varepsilon & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\Sigma}_\varepsilon \end{pmatrix}.$$

From (2), (3) and (4) it follows that

$$\boldsymbol{Y}_j = \boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j + \boldsymbol{X}_j\boldsymbol{\lambda}_j + \boldsymbol{\varepsilon}_j$$

and

$$\boldsymbol{V}_j = \boldsymbol{X}_j\boldsymbol{\lambda}_j + \boldsymbol{\varepsilon}_j \sim \mathrm{N}_{pn(j)}\left(\mathbf{0}, \boldsymbol{X}_j\boldsymbol{\Sigma}_\lambda\boldsymbol{X}_j' + \boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon\right).$$

Reinsel (1984) studied a case where $n = n(j)$, $\boldsymbol{X} = \boldsymbol{X}_j$ and $\boldsymbol{X} = \boldsymbol{\Pi} \otimes \boldsymbol{I}_n$, where $\boldsymbol{\Pi}$ is a $p \times a$ matrix and $q = an$. Furthermore, Reinsel (1985) dealed with a model of $p = 1$ and $\boldsymbol{\Sigma}_\varepsilon = \sigma_\varepsilon^2$.

In Section 2 we show estimators of $\boldsymbol{\beta}_j$ when the other parameters are supposed to be known. A likelihood function of the model is given in Section 3. Moreover we consider some asumptions when maximum likelihood estimators of $\boldsymbol{\Sigma}_\varepsilon$, $\boldsymbol{B}$ and $\boldsymbol{\Sigma}_\lambda$ are explicitly expressed.

2 Estimator of $\boldsymbol{\beta}_j$

Here we suppose that $\boldsymbol{\Sigma}_\varepsilon$, $\boldsymbol{B}$ and $\boldsymbol{\Sigma}_\lambda$ are known. We put

$$\tilde{\boldsymbol{\beta}}_j = \left[\boldsymbol{X}_j'\left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right)\boldsymbol{X}_j\right]^{-1}\boldsymbol{X}_j'\left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right)\boldsymbol{Y}_j$$

that is a generalized least squares estimator of $\boldsymbol{\beta}_j$ of the model (3) and (4). Then an estimator of $\boldsymbol{\beta}_j$ is given by

$$\begin{aligned}
& \boldsymbol{\beta}_j^*\left(\boldsymbol{\Sigma}_\varepsilon, \boldsymbol{B}, \boldsymbol{\Sigma}_\lambda\right) \\
= & \left[\boldsymbol{X}_j'\left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right)\boldsymbol{X}_j + \boldsymbol{\Sigma}_\lambda^{-1}\right]^{-1}\left[\boldsymbol{X}_j'\left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right)\boldsymbol{X}_j\tilde{\boldsymbol{\beta}}_j + \boldsymbol{\Sigma}_\lambda^{-1}\boldsymbol{B}\boldsymbol{a}_j\right] \\
= & \tilde{\boldsymbol{\beta}}_j - \left[\boldsymbol{X}_j'\left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right)\boldsymbol{X}_j\right]^{-1}\left\{\left[\boldsymbol{X}_j'\left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right)\boldsymbol{X}_j\right]^{-1} + \boldsymbol{\Sigma}_\lambda\right\}^{-1}\left(\tilde{\boldsymbol{\beta}}_j - \boldsymbol{B}\boldsymbol{a}_j\right) \\
= & \boldsymbol{B}\boldsymbol{a}_j - \boldsymbol{\Sigma}_\lambda\left\{\left[\boldsymbol{X}_j'\left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right)\boldsymbol{X}_j\right]^{-1} + \boldsymbol{\Sigma}_\lambda\right\}^{-1}\left(\boldsymbol{B}\boldsymbol{a}_j - \tilde{\boldsymbol{\beta}}_j\right) \\
= & \mathcal{E}\left(\boldsymbol{\beta}_j \,\middle|\, \boldsymbol{Y}_j\right) \\
= & \mathcal{E}\left(\boldsymbol{\beta}_j \,\middle|\, \tilde{\boldsymbol{\beta}}_j\right),
\end{aligned}$$

where $\mathcal{E}\left(\boldsymbol{\beta}_j \,\middle|\, \boldsymbol{Y}_j\right)$ and $\mathcal{E}\left(\boldsymbol{\beta}_j \,\middle|\, \tilde{\boldsymbol{\beta}}_j\right)$ are conditional expectations of $\boldsymbol{\beta}_j$ on $\boldsymbol{Y}_j$ and $\tilde{\boldsymbol{\beta}}_j$, respectively.

Furthermore, a mean squared error matrix of $\boldsymbol{\beta}_j^*\left(\boldsymbol{\Sigma}_\varepsilon, \boldsymbol{B}, \boldsymbol{\Sigma}_\lambda\right)$ is the conditional covariance matrix $\mathcal{C}ov\left[\boldsymbol{\beta}_j^*\left(\boldsymbol{\Sigma}_\varepsilon, \boldsymbol{B}, \boldsymbol{\Sigma}_\lambda\right) \,\middle|\, \boldsymbol{Y}_j\right]$ of $\boldsymbol{\beta}_j$ on $\boldsymbol{Y}_j$, where

$$\begin{aligned}
& \mathcal{C}ov\left[\boldsymbol{\beta}_j^*\left(\boldsymbol{\Sigma}_\varepsilon, \boldsymbol{B}, \boldsymbol{\Sigma}_\lambda\right) \,\middle|\, \boldsymbol{Y}_j\right] \\
= & \left[\boldsymbol{X}_j'\left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right)\boldsymbol{X}_j + \boldsymbol{\Sigma}_\lambda^{-1}\right]^{-1} \\
= & \boldsymbol{\Sigma}_\lambda - \boldsymbol{\Sigma}_\lambda\left\{\left[\boldsymbol{X}_j'\left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right)\boldsymbol{X}_j\right]^{-1} + \boldsymbol{\Sigma}_\lambda\right\}^{-1}\boldsymbol{\Sigma}_\lambda \\
= & \left[\boldsymbol{X}_j'\left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right)\boldsymbol{X}_j\right]^{-1}
\end{aligned}$$

$$
- \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1} \right) \boldsymbol{X}_j \right]^{-1} \left\{ \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1} \right) \boldsymbol{X}_j \right]^{-1} + \boldsymbol{\Sigma}_\lambda \right\}^{-1}
\times \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1} \right) \boldsymbol{X}_j \right]^{-1}.
$$

Note that

$$
\mathcal{C}ov \left(\tilde{\boldsymbol{\beta}}_j \,\middle|\, \boldsymbol{\beta}_j \right) = \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1} \right) \boldsymbol{X}_j \right]^{-1}.
$$

3 Maximum likelihood estimation

We consider estimation of $\boldsymbol{\Sigma}_\varepsilon$, $\boldsymbol{B}$ and $\boldsymbol{\Sigma}_\lambda$ by the method of maximum likelihood.

3.1 Likelihood function

A likelihood function of $\boldsymbol{\Sigma}_\varepsilon$, $\boldsymbol{B}$ and $\boldsymbol{\Sigma}_\lambda$ subject to $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_N$ is given by

$$
\begin{aligned}
& L \left(\boldsymbol{\Sigma}_\varepsilon, \boldsymbol{B}, \boldsymbol{\Sigma}_\lambda \,\middle|\, \boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_N \right) \\
= & (2\pi)^{-\frac{pT}{2}} \prod_{j=1}^{N} \left| \boldsymbol{X}_j \boldsymbol{\Sigma}_\lambda \boldsymbol{X}_j' + \boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon \right|^{-\frac{1}{2}} \\
& \times \exp \left[-\frac{1}{2} \sum_{j=1}^{N} \left(\boldsymbol{Y}_j - \boldsymbol{X}_j \boldsymbol{B} \boldsymbol{a}_j \right)' \left(\boldsymbol{X}_j \boldsymbol{\Sigma}_\lambda \boldsymbol{X}_j' + \boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon \right)^{-1} \left(\boldsymbol{Y}_j - \boldsymbol{X}_j \boldsymbol{B} \boldsymbol{a}_j \right) \right],
\end{aligned} \tag{5}
$$

where $T = \sum_{j=1}^{N} n(j)$.

Moreover the logarithm of the likelihood function $L \left(\boldsymbol{\Sigma}_\varepsilon, \boldsymbol{B}, \boldsymbol{\Sigma}_\lambda \,\middle|\, \boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_N \right)$ is given by

$$
\begin{aligned}
& \log \left[L \left(\boldsymbol{\Sigma}_\varepsilon, \boldsymbol{B}, \boldsymbol{\Sigma}_\lambda \,\middle|\, \boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_N \right) \right] \\
= & -\frac{pT}{2} \log(2\pi) - \frac{T}{2} \log \left| \boldsymbol{\Sigma}_\varepsilon \right| - \frac{1}{2} \sum_{j=1}^{N} \log \left| \boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1} \right) \boldsymbol{X}_j \right| - \frac{1}{2} \sum_{j=1}^{n} \log \left| \boldsymbol{\Omega}_j \right| \\
& - \frac{1}{2} \sum_{j=1}^{n} \left(\boldsymbol{Y}_j - \boldsymbol{X}_j \boldsymbol{B} \boldsymbol{a}_j \right)' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1} \right) \left(\boldsymbol{Y}_j - \boldsymbol{X}_j \boldsymbol{B} \boldsymbol{a}_j \right) \\
& + \frac{1}{2} \sum_{j=1}^{N} \left(\boldsymbol{X}_j \tilde{\boldsymbol{\beta}}_j - \boldsymbol{X}_j \boldsymbol{B} \boldsymbol{a}_j \right)' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1} \right) \left(\boldsymbol{X}_j \tilde{\boldsymbol{\beta}}_j - \boldsymbol{X}_j \boldsymbol{B} \boldsymbol{a}_j \right) \\
& - \frac{1}{2} \sum_{j=1}^{N} \left(\tilde{\boldsymbol{\beta}}_j - \boldsymbol{B} \boldsymbol{a}_j \right)' \boldsymbol{\Omega}_j^{-1} \left(\tilde{\boldsymbol{\beta}}_j - \boldsymbol{B} \boldsymbol{a}_j \right).
\end{aligned} \tag{6}
$$

See Appendix. Moreover, we have

$$\begin{aligned}&\max_{\boldsymbol{\Sigma}_\varepsilon,\boldsymbol{B},\boldsymbol{\Sigma}_\lambda} \log\left[L\left(\boldsymbol{\Sigma}_\varepsilon,\boldsymbol{B},\boldsymbol{\Sigma}_\lambda \,\middle|\, \boldsymbol{Y}_1,\boldsymbol{Y}_2,\cdots,\boldsymbol{Y}_N\right)\right]\\ =\ &\max_{\boldsymbol{\Sigma}_\varepsilon,\boldsymbol{\Sigma}_\lambda} \log\left[L\left(\boldsymbol{\Sigma}_\varepsilon,\tilde{\boldsymbol{B}},\boldsymbol{\Sigma}_\lambda \,\middle|\, \boldsymbol{Y}_1,\boldsymbol{Y}_2,\cdots,\boldsymbol{Y}_N\right)\right].\end{aligned}$$

3.2 Maximum likelihood estimator of $\boldsymbol{B}$ when $\boldsymbol{\Sigma}_\varepsilon$ and $\boldsymbol{\Sigma}_\lambda$ are known

Here we assume that $\boldsymbol{\Sigma}_\varepsilon$ and $\boldsymbol{\Sigma}_\lambda$ are known. Then, a maximum likelihood estimator of $\boldsymbol{B}$ when $\boldsymbol{Y}_1,\boldsymbol{Y}_2,\ldots,\boldsymbol{Y}_N$ is equivalent for a maximum likelihood estimator of $\boldsymbol{B}$ when $\tilde{\boldsymbol{\beta}}_1,\tilde{\boldsymbol{\beta}}_2,\ldots,\tilde{\boldsymbol{\beta}}_N$.

Let $\boldsymbol{u}_j, j=1,2,\ldots,N$ be independently and identically distributed as a normal distribution $\mathrm{N}_q\left(\boldsymbol{0},\boldsymbol{\Omega}_j\right)$, where

$$\boldsymbol{\Omega}_j=\left[\boldsymbol{X}_j'\left(\boldsymbol{I}_{n(j)}\otimes\boldsymbol{\Sigma}_\varepsilon^{-1}\right)\boldsymbol{X}_j\right]^{-1}+\boldsymbol{\Sigma}_\lambda.$$

Further, we put $\boldsymbol{B}=(\boldsymbol{b}_1,\boldsymbol{b}_2,\ldots,\boldsymbol{b}_r)$, where $\boldsymbol{b}_j$'s are $q\times 1$ vectors, a $r\times N$ matrix $\boldsymbol{A}=(\boldsymbol{a}_1,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_N)$ and a $q\times N$ matrix $\boldsymbol{U}=(\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_N)$. Then

$$\left(\tilde{\boldsymbol{\beta}}_1,\tilde{\boldsymbol{\beta}}_2,\ldots,\tilde{\boldsymbol{\beta}}_N\right)=\boldsymbol{B}\boldsymbol{A}+\boldsymbol{U}.$$

Hence, we have a model

$$\begin{pmatrix}\tilde{\boldsymbol{\beta}}_1\\ \tilde{\boldsymbol{\beta}}_2\\ \vdots\\ \tilde{\boldsymbol{\beta}}_N\end{pmatrix}=\left(\boldsymbol{A}'\otimes\boldsymbol{I}_q\right)\mathrm{vec}\left(\boldsymbol{B}\right)+\mathrm{vec}\left(\boldsymbol{U}\right) \tag{7}$$

and

$$\mathrm{vec}\left(\boldsymbol{U}\right)\sim\mathrm{N}_{qN}\left[\boldsymbol{0},\mathrm{Diag}\left(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2,\ldots,\boldsymbol{\Omega}_N\right)\right], \tag{8}$$

where $\boldsymbol{I}_q$ is an identity matrix of order q and 'vec' and 'Diag' respectively mean operators as follows:

$$\mathrm{vec}\,(\boldsymbol{B}) = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_r \end{pmatrix}, \mathrm{vec}\,(\boldsymbol{U}) = \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_N \end{pmatrix}$$

and

$$\mathrm{Diag}\,(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \ldots, \boldsymbol{\Omega}_N) = \begin{pmatrix} \boldsymbol{\Omega}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\Omega}_N \end{pmatrix}.$$

Note that $\mathrm{vec}\,(\boldsymbol{CDE}) = (\boldsymbol{E}' \otimes \boldsymbol{C})\,\mathrm{vec}\,(\boldsymbol{D})$, where $\boldsymbol{C}$, $\boldsymbol{D}$ and $\boldsymbol{E}$ are $c \times d$, $d \times e$ and $e \times f$ matrices, respectively.

In the model (7) and (8), a generalized least squares estimator of $\mathrm{vec}\,(\boldsymbol{B})$ is given by

$$\begin{aligned} \widetilde{\mathrm{vec}\,(\boldsymbol{B})} &= \left\{ (\boldsymbol{A} \otimes \boldsymbol{I}_q) \left[\mathrm{Diag}\,(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \ldots, \boldsymbol{\Omega}_N) \right]^{-1} (\boldsymbol{A}' \otimes \boldsymbol{I}_q) \right\}^{-1} \\ &\quad \times (\boldsymbol{A} \otimes \boldsymbol{I}_q) \left[\mathrm{Diag}\,(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \ldots, \boldsymbol{\Omega}_N) \right]^{-1} \begin{pmatrix} \tilde{\boldsymbol{\beta}}_1 \\ \tilde{\boldsymbol{\beta}}_2 \\ \vdots \\ \tilde{\boldsymbol{\beta}}_N \end{pmatrix} \\ &= \left\{ \sum_{j=1}^{N} \left[\left(\boldsymbol{a}_j \boldsymbol{a}_j' \right) \otimes \boldsymbol{\Omega}_j^{-1} \right] \right\}^{-1} \sum_{j=1}^{N} \left(\boldsymbol{a}_j \otimes \boldsymbol{\Omega}_j^{-1} \right) \tilde{\boldsymbol{\beta}}_j \end{aligned}$$

that is a maximum likelihood estimator of $\mathrm{vec}\,(\boldsymbol{B})$. Therefore a maximum likelihood estimator $\tilde{\boldsymbol{B}}$ of $\boldsymbol{B}$ is obtaind by detaching the 'vec' operator from $\widetilde{\mathrm{vec}\,(\boldsymbol{B})}$.

3.3 Maximum likelihood estimators of $\boldsymbol{\Sigma}_\varepsilon$, $\boldsymbol{B}$ and $\boldsymbol{\Sigma}_\lambda$

Usually not only $\boldsymbol{B}$ but also $\boldsymbol{\Sigma}_\varepsilon$ and $\boldsymbol{\Sigma}_\lambda$ are unknown. In order to estimate $\boldsymbol{\Sigma}_\varepsilon$ and $\boldsymbol{\Sigma}_\lambda$, we replace $\boldsymbol{B}$ by $\tilde{\boldsymbol{B}}$ in the likelihood function and maximize it with respect to $\boldsymbol{\Sigma}_\varepsilon$ and $\boldsymbol{\Sigma}_\lambda$. However, it is difficult to derive explicit expressions of the maximum likelihood estimators in the generalized case. Here we shall consider some assumptions on the $\boldsymbol{\Sigma}_\varepsilon$ and $\boldsymbol{X}_j$.

3.3.1 Estimation in a case where $\boldsymbol{\Sigma}_\varepsilon = \sigma_\varepsilon^2 \boldsymbol{I}_p$

In this case, it holds that

$$\boldsymbol{\Omega}_j = \sigma_\varepsilon^2 \left(\boldsymbol{X}_j' \boldsymbol{X}_j \right)^{-1} + \boldsymbol{\Sigma}_\lambda$$

and $\tilde{\boldsymbol{\beta}}_j$ is equal to an ordinary least squares estimator, that is,

$$\tilde{\boldsymbol{\beta}}_j = \hat{\boldsymbol{\beta}}_j = \left(\boldsymbol{X}_j' \boldsymbol{X}_j \right)^{-1} \boldsymbol{X}_j' \boldsymbol{Y}_j.$$

Therefore a log likelihood function is given by

$$\begin{aligned} & \log \left[L \left(\sigma_\varepsilon^2, \boldsymbol{B}, \boldsymbol{\Sigma}_\lambda \,\middle|\, \boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_N \right) \right] \\ = \; & -\frac{pT}{2} \log(2\pi) - \frac{pT - qN}{2} \log \sigma_\varepsilon^2 - \frac{1}{2} \sum_{j=1}^{N} \log \left| \boldsymbol{X}_j' \boldsymbol{X}_j \right| - \frac{1}{2} \sum_{j=1}^{n} \log \left| \boldsymbol{\Omega}_j \right| \\ & - \frac{1}{2\sigma_\varepsilon^2} \sum_{j=1}^{n} (\boldsymbol{Y}_j - \boldsymbol{X}_j \boldsymbol{B} \boldsymbol{a}_j)' (\boldsymbol{Y}_j - \boldsymbol{X}_j \boldsymbol{B} \boldsymbol{a}_j) \\ & + \frac{1}{2\sigma_\varepsilon^2} \sum_{j=1}^{N} \left(\boldsymbol{X}_j \hat{\boldsymbol{\beta}}_j - \boldsymbol{X}_j \boldsymbol{B} \boldsymbol{a}_j \right)' \left(\boldsymbol{X}_j \hat{\boldsymbol{\beta}}_j - \boldsymbol{X}_j \boldsymbol{B} \boldsymbol{a}_j \right) \\ & - \frac{1}{2} \sum_{j=1}^{N} \left(\hat{\boldsymbol{\beta}}_j - \boldsymbol{B} \boldsymbol{a}_j \right)' \boldsymbol{\Omega}_j^{-1} \left(\hat{\boldsymbol{\beta}}_j - \boldsymbol{B} \boldsymbol{a}_j \right). \end{aligned}$$

Then explicit expressions of maximum likelihood estimators of the parameters are not obtained.

3.3.2 Estimation in a case where $n(j) = n$ and $\boldsymbol{X}_j = \boldsymbol{X}$

Suppose that $n(j) = n$ and $\boldsymbol{X}_j = \boldsymbol{X}$. Then we have

$$\boldsymbol{\Omega}_j = \boldsymbol{\Omega} = \left[\boldsymbol{X}' \left(\boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_\varepsilon^{-1} \right) \boldsymbol{X} \right]^{-1} + \boldsymbol{\Sigma}_\lambda.$$

Thus it holds that

$$\begin{aligned} & \log \left[L \left(\boldsymbol{\Sigma}_\varepsilon, \boldsymbol{B}, \boldsymbol{\Sigma}_\lambda \,\middle|\, \boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_N \right) \right] \\ = \; & -\frac{pT}{2} \log(2\pi) - \frac{T}{2} \log \left| \boldsymbol{\Sigma}_\varepsilon \right| - \frac{N}{2} \log \left| \boldsymbol{X}' \left(\boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_\varepsilon^{-1} \right) \boldsymbol{X} \right| - \frac{N}{2} \log \left| \boldsymbol{\Omega} \right| \\ & - \frac{1}{2} \sum_{j=1}^{n} (\boldsymbol{Y}_j - \boldsymbol{X} \boldsymbol{B} \boldsymbol{a}_j)' \left(\boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_\varepsilon^{-1} \right) (\boldsymbol{Y}_j - \boldsymbol{X} \boldsymbol{B} \boldsymbol{a}_j) \end{aligned}$$

$$+\frac{1}{2}\sum_{j=1}^{N}\left(\boldsymbol{X}\tilde{\boldsymbol{\beta}}_j-\boldsymbol{X}\boldsymbol{B}\boldsymbol{a}_j\right)'\left(\boldsymbol{I}_n\otimes\boldsymbol{\Sigma}_\varepsilon^{-1}\right)\left(\boldsymbol{X}\tilde{\boldsymbol{\beta}}_j-\boldsymbol{X}\boldsymbol{B}\boldsymbol{a}_j\right)$$
$$-\frac{1}{2}\sum_{j=1}^{N}\left(\tilde{\boldsymbol{\beta}}_j-\boldsymbol{B}\boldsymbol{a}_j\right)'\boldsymbol{\Omega}^{-1}\left(\tilde{\boldsymbol{\beta}}_j-\boldsymbol{B}\boldsymbol{a}_j\right).$$

Then the log likelihood function of $\boldsymbol{\Sigma}_\varepsilon$, $\boldsymbol{B}$ and $\boldsymbol{\Sigma}_\lambda$ is equivalent to the log likelihood function of $\boldsymbol{\Sigma}_\varepsilon$, $\boldsymbol{B}$ and $\boldsymbol{\Omega}$.

We have estimators

$$\tilde{\boldsymbol{B}}=\sum_{j=1}^{N}\tilde{\boldsymbol{\beta}}_j\boldsymbol{a}_j'\left(\boldsymbol{A}\boldsymbol{A}'\right)^{-1}=\left(\tilde{\boldsymbol{\beta}}_1,\tilde{\boldsymbol{\beta}}_2,\ldots,\tilde{\boldsymbol{\beta}}_N\right)\boldsymbol{A}'\left(\boldsymbol{A}\boldsymbol{A}'\right)^{-1}$$

and

$$\tilde{\boldsymbol{\Omega}}=\frac{1}{N}\sum_{j=1}^{N}\left(\tilde{\boldsymbol{\beta}}_j-\tilde{\boldsymbol{B}}\boldsymbol{a}_j\right)\left(\tilde{\boldsymbol{\beta}}_j-\tilde{\boldsymbol{B}}\boldsymbol{a}_j\right)'.$$

Both of $\tilde{\boldsymbol{B}}$ and $\tilde{\boldsymbol{\Omega}}$ depend on $\boldsymbol{\Sigma}_\varepsilon$. Then we estimate $\boldsymbol{\Sigma}_\varepsilon$ by miximaizing the log likelihood function after replacing $\boldsymbol{B}$ and $\boldsymbol{\Omega}$ by $\tilde{\boldsymbol{B}}$ and $\tilde{\boldsymbol{\Omega}}$ respectively. An estimator $\tilde{\boldsymbol{\Sigma}}_\varepsilon$ of $\boldsymbol{\Sigma}_\varepsilon$ is not explicitly expressed. We estimate $\boldsymbol{\Sigma}_\lambda$ with

$$\tilde{\boldsymbol{\Sigma}}_\lambda=\tilde{\boldsymbol{\Omega}}-\left[\boldsymbol{X}'\left(\boldsymbol{I}_n\otimes\tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1}\right)\boldsymbol{X}\right]^{-1}$$

subject to nonsingular $\tilde{\boldsymbol{\Sigma}}_\varepsilon$.

3.3.3 Estimation in a case where $n(j)=n$, $\boldsymbol{X}_j=\boldsymbol{X}$ and $\boldsymbol{\Sigma}_\varepsilon=\sigma_\varepsilon^2\boldsymbol{I}_p$

We suppose that $n(j)=n$, $\boldsymbol{X}_j=\boldsymbol{X}$ and $\boldsymbol{\Sigma}_\varepsilon=\sigma_\varepsilon^2\boldsymbol{I}_p$. It holds that

$$\boldsymbol{\Omega}_j=\sigma_\varepsilon^2\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}+\boldsymbol{\Sigma}_\lambda$$

and $\tilde{\boldsymbol{\beta}}_j=\hat{\boldsymbol{\beta}}_j$. Hence a log likelihood function is given by

$$\begin{aligned}
&\log\left[L\left(\sigma_\varepsilon^2,\boldsymbol{B},\boldsymbol{\Sigma}_\lambda\;\middle|\;\boldsymbol{Y}_1,\boldsymbol{Y}_2,\cdots,\boldsymbol{Y}_N\right)\right]\\
=\;&-\frac{pT}{2}\log\left(2\pi\right)-\frac{pT-qN}{2}\log\sigma_\varepsilon^2-\frac{N}{2}\log\left|\,\boldsymbol{X}'\boldsymbol{X}\,\right|-\frac{N}{2}\log\left|\,\boldsymbol{\Omega}\,\right|\\
&-\frac{1}{2\sigma_\varepsilon^2}\sum_{j=1}^{n}\left(\boldsymbol{Y}_j-\boldsymbol{X}\boldsymbol{B}\boldsymbol{a}_j\right)'\left(\boldsymbol{Y}_j-\boldsymbol{X}\boldsymbol{B}\boldsymbol{a}_j\right)\\
&+\frac{1}{2\sigma_\varepsilon^2}\sum_{j=1}^{N}\left(\boldsymbol{X}\hat{\boldsymbol{\beta}}_j-\boldsymbol{X}\boldsymbol{B}\boldsymbol{a}_j\right)'\left(\boldsymbol{X}\hat{\boldsymbol{\beta}}_j-\boldsymbol{X}\boldsymbol{B}\boldsymbol{a}_j\right)\\
&-\frac{1}{2}\sum_{j=1}^{N}\left(\hat{\boldsymbol{\beta}}_j-\boldsymbol{B}\boldsymbol{a}_j\right)'\boldsymbol{\Omega}^{-1}\left(\hat{\boldsymbol{\beta}}_j-\boldsymbol{B}\boldsymbol{a}_j\right).
\end{aligned}$$

We estimate $\boldsymbol{B}$ and $\boldsymbol{\Omega}$ with

$$\hat{\boldsymbol{B}} = \sum_{j=1}^{N} \hat{\boldsymbol{\beta}}_j \boldsymbol{a}'_j \left(\boldsymbol{A}\boldsymbol{A}'\right)^{-1} = \left(\hat{\boldsymbol{\beta}}_1, \hat{\boldsymbol{\beta}}_2, \ldots, \hat{\boldsymbol{\beta}}_N\right) \boldsymbol{A}' \left(\boldsymbol{A}\boldsymbol{A}'\right)^{-1}$$

and

$$\hat{\boldsymbol{\Omega}} = \frac{1}{N} \sum_{j=1}^{N} \left(\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{B}}\boldsymbol{a}_j\right) \left(\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{B}}\boldsymbol{a}_j\right)',$$

respectively. These estimators do not depend on σ_ε^2. Then we estimate σ_ε^2 with

$$\begin{aligned} \hat{\sigma}_\varepsilon^2 \;\; = \;\; & \frac{1}{pT - qN} \sum_{j=1}^{N} \left[\left(\boldsymbol{Y}_j - \boldsymbol{X}\hat{\boldsymbol{B}}\boldsymbol{a}_j\right)' \left(\boldsymbol{Y}_j - \boldsymbol{X}\hat{\boldsymbol{B}}\boldsymbol{a}_j\right) \right. \\ & \left. - \left(\boldsymbol{X}\hat{\boldsymbol{\beta}}_j - \boldsymbol{X}\hat{\boldsymbol{B}}\boldsymbol{a}_j\right)' \left(\boldsymbol{X}\hat{\boldsymbol{\beta}}_j - \boldsymbol{X}\hat{\boldsymbol{B}}\boldsymbol{a}_j\right) \right]. \end{aligned}$$

At last an estimator of $\boldsymbol{\Sigma}_\lambda$ is given by

$$\hat{\boldsymbol{\Sigma}}_\lambda = \hat{\boldsymbol{\Omega}} - \hat{\sigma}_\varepsilon^2 \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}.$$

Appendix

Here we show (6). In (5) it holds that

$$\begin{aligned} & \prod_{j=1}^{N} \left| \boldsymbol{X}_j \boldsymbol{\Sigma}_\lambda \boldsymbol{X}'_j + \boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon \right|^{-\frac{1}{2}} \\ = & \prod_{j=1}^{N} \left| \boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon \right|^{-\frac{1}{2}} \left| \boldsymbol{I}_{pn(j)} + \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j \boldsymbol{\Sigma}_\lambda \boldsymbol{X}'_j \right|^{-\frac{1}{2}} \\ = & \left| \boldsymbol{\Sigma}_\varepsilon \right|^{-\frac{T}{2}} \prod_{j=1}^{N} \left| \boldsymbol{I}_q + \boldsymbol{\Sigma}_\lambda \boldsymbol{X}'_j \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j \right|^{-\frac{1}{2}} \\ = & \left| \boldsymbol{\Sigma}_\varepsilon \right|^{-\frac{T}{2}} \prod_{j=1}^{N} \left| \boldsymbol{X}'_j \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j \right|^{-\frac{1}{2}} \left| \left[\boldsymbol{X}'_j \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j\right]^{-1} + \boldsymbol{\Sigma}_\lambda \right|^{-\frac{1}{2}} \\ = & \left| \boldsymbol{\Sigma}_\varepsilon \right|^{-\frac{T}{2}} \prod_{j=1}^{N} \left| \boldsymbol{X}'_j \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j \right|^{-\frac{1}{2}} \left| \boldsymbol{\Omega}_j \right|^{-\frac{1}{2}}. \qquad (9) \end{aligned}$$

The second equation follows from $| \boldsymbol{I}_c + \boldsymbol{C}\boldsymbol{D} | = | \boldsymbol{I}_d + \boldsymbol{D}\boldsymbol{C} |$, where $\boldsymbol{C}$ and $\boldsymbol{D}$ are a $c \times d$ matrix and a $d \times c$ matrix, respectively.

Further we have

$$
\begin{aligned}
&\left(\boldsymbol{X}_j \boldsymbol{\Sigma}_\lambda \boldsymbol{X}_j' + \boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon\right)^{-1} \\
= \;& \boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1} - \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j + \boldsymbol{\Sigma}_\lambda^{-1}\right]^{-1} \boldsymbol{X}_j' \\
& \qquad \times \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \\
= \;& \boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1} - \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j\right]^{-1} \boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \\
& + \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j\right]^{-1} \boldsymbol{\Omega}_j^{-1} \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j\right]^{-1} \boldsymbol{X}_j' \\
& \times \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right).
\end{aligned}
$$

The first equation follows from

$$
(\boldsymbol{C} + \boldsymbol{E}\boldsymbol{D}\boldsymbol{E}')^{-1} = \boldsymbol{C}^{-1} - \boldsymbol{C}^{-1}\boldsymbol{E}\left(\boldsymbol{E}'\boldsymbol{C}^{-1}\boldsymbol{E} + \boldsymbol{D}^{-1}\right)^{-1}\boldsymbol{E}'\boldsymbol{C}^{-1},
$$

where $\boldsymbol{C}$, $\boldsymbol{D}$ and $\boldsymbol{E}$ are a $c \times c$ nonsingular matrix, a $d \times d$ nonsingular matrix and $c \times d$ matrix, respectively. See Rao (1973, p.33). Note that

$$
\begin{aligned}
& \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j + \boldsymbol{\Sigma}_\lambda^{-1}\right]^{-1} \\
= \;& \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j\right]^{-1} \\
& - \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j\right]^{-1} \boldsymbol{\Omega}_j^{-1} \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j\right]^{-1}.
\end{aligned}
$$

Then, it is derived that

$$
\begin{aligned}
& \sum_{j=1}^{N} (\boldsymbol{Y}_j - \boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j)' \left(\boldsymbol{X}_j \boldsymbol{\Sigma}_\lambda \boldsymbol{X}_j' + \boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon\right)^{-1} (\boldsymbol{Y}_j - \boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j) \\
= \;& \sum_{j=1}^{N} (\boldsymbol{Y}_j - \boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j)' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) (\boldsymbol{Y}_j - \boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j) \\
& - \sum_{j=1}^{N} (\boldsymbol{Y}_j - \boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j)' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j\right]^{-1} \\
& \qquad \times \boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) (\boldsymbol{Y}_j - \boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j) \\
& + \sum_{j=1}^{N} (\boldsymbol{Y}_j - \boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j)' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j\right]^{-1} \boldsymbol{X}_j' \\
& \qquad \times \boldsymbol{\Omega}_j^{-1} \boldsymbol{X}_j \left[\boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) \boldsymbol{X}_j\right]^{-1} \boldsymbol{X}_j' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) (\boldsymbol{Y}_j - \boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j) \\
= \;& \sum_{j=1}^{n} (\boldsymbol{Y}_j - \boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j)' \left(\boldsymbol{I}_{n(j)} \otimes \boldsymbol{\Sigma}_\varepsilon^{-1}\right) (\boldsymbol{Y}_j - \boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j)
\end{aligned}
$$

$$-\sum_{j=1}^{N}\left(\boldsymbol{X}_j\tilde{\boldsymbol{\beta}}_j-\boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j\right)'\left(\boldsymbol{I}_{n(j)}\otimes\boldsymbol{\Sigma}_{\varepsilon}^{-1}\right)\left(\boldsymbol{X}_j\tilde{\boldsymbol{\beta}}_j-\boldsymbol{X}_j\boldsymbol{B}\boldsymbol{a}_j\right)$$
$$+\sum_{j=1}^{N}\left(\tilde{\boldsymbol{\beta}}_j-\boldsymbol{B}\boldsymbol{a}_j\right)'\boldsymbol{\Omega}_j^{-1}\left(\tilde{\boldsymbol{\beta}}_j-\boldsymbol{B}\boldsymbol{a}_j\right). \tag{10}$$

Thus (6) is obtained by using (9) and (10).

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